ANALYTIC TORSION FOR QUATERNIONIC MANIFOLDS AND RELATED TOPICS

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Abstract. In this paper we show that the Ray-Singer complex analytic torsion is trivial for even dimensional Calabi-Yau manifolds.

Then we define the quaternionic analytic torsion for quaternionic manifolds and prove that they are metric independent.

In dimension four, the quaternionic analytic torsion equals to the self-dual analytic torsion. For higher dimensional manifolds, the self-dual analytic torsion is a conformal invariant.

1. Introduction

In the seminal papers ($[RS1]$, $[RS2]$), Ray and Singer defined analytic torsions for real and complex manifolds as determinants of the deRham complex and Dolbeault complex respectively. As an analog to the vanishing of real analytic torsions for even dimensional manifolds, we prove the following vanishing result for complex analytic torsions.

Theorem 1. *If M is an even dimensional Calabi-Yau manifold, then*

$$\log \tau_{\mathbb{C}}(M,V)=0,$$

where $\tau_{\mathbb{C}}(M,V)$ is the complex analytic torsion for M with coefficient in the unitary flat bundle V.

In particular the complex analytic torsion is always trivial for HyperKähler manifolds. Naturally we look for refinement of the complex analytic torsion. In section 3, we define a quaternionic analytic torsion $\tau_{\mathbb{H}}(M,V)$ for quaternionic manifolds.

Recall that a $4n$-dimensional manifold M is called a quaternionic manifold if its tangent bundle admits a torsion-free $GL(n,\mathbb{H})\,Sp(1)$ connection. Let E and H denote the associated bundles to the frame bundle of M with respect to standard representations of $GL(n,\mathbb{H})$ and $Sp(1)$.

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Theorem 2. Let V be a unitary flat bundle over a quaternionic manifold M. For any compatible metric on M, we define $\tau_{\mathbb{H}}(M,V)$ to be the regularized determinant of the following elliptic complex:

$$0 \to A^0 \xrightarrow{D} A^1 \xrightarrow{D} \cdots \xrightarrow{D} A^{2n} \to 0,$$

where $A^k = \Gamma(M, \Lambda^k E \otimes S^k H \otimes V)$.

Then the ratio $\tau_{\mathbb{H}}(M,V_1)/\tau_{\mathbb{H}}(M,V_2)$ is independent of the choice of the compatible metric on M provided that their corresponding complexes have trivial cohomology groups.

The above complex was introduced by S. M. Salamon ([S1]).

Remark. The assumption about trivial cohomology groups is not essential. Otherwise we can write down the variation of $\tau_{\mathbb{H}}$ in terms of the change of volume forms on the determinant of cohomology groups.

Notice that every Riemannian metric on a four manifold M determines a quaternionic structure on M. The above elliptic complex is the same as the self-dual complex:

$$0 \to \Omega^0(M,V) \xrightarrow{d} \Omega^1(M,V) \xrightarrow{\sqrt{2}P_+d} \Omega^2_+(M,V) \to 0.$$

Because conformally equivalent metrics determine the same quaternionic structure on M, above theorem shows that the regularized determinant of the self-dual complex of a four manifold depends only on the conformal class of the metric. More generally, the following is true:

Theorem 3. We assume that M is $4n$-dimensional closed Riemannian manifold and V is an orthogonal flat vector bundle over M. Let

$$\tau_{\mathrm{SD}}(M,V) = \sum_{q=0}^{2n-1} (-1)^{q+1}\, q \log\det(\triangle_q) - n\log\det(\triangle_{2n}).$$

Then

(i) $\tau_{\mathrm{SD}}(M,V)$ is the regularized determinant of the following self-dual complex:

$$0 \to \Omega^0(M,V) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{2n-1}(M,V) \xrightarrow{\sqrt{2}P_+d} \Omega^{2n}_+(M,V) \to 0.$$

(ii) $\tau_{\mathrm{SD}}(M,V_1)/\tau_{\mathrm{SD}}(M,V_2)$ depends only on the conformal class of the Riemannian metric provided that the above self-dual complex has trivial cohomology groups.

A similar result on conformal invariance of self-dual analytic torsion has been known to J. Cheeger for many years. After this paper is finished, the author is informed by Bismut that he also noticed the result of theorem one before by using Serre dualit y for Quillen metrics.

2. Vanishing of Complex Analytic Torsions

Ray and Singer [RS1] defined the real analytic torsion $\tau_{\mathbb{R}}(M)$ for a compact oriented Riemannian manifold M of dimension n, using zeta functions of Laplacian and showed that they are metric independent. Moreover, $\tau_{\mathbb{R}}$ is trivial when M is an even dimensional manifold. In this section, we shall prove a complex analog of their theorem. To begin, we first recall their construction of the real analytic torsion,

$$\tau_{\mathbb{R}}(M,V) = \exp\left(\frac{1}{2}\sum_{q=0}^{n}(-1)^q q\zeta'_{q,V}(0)\right),$$

where V denotes both an orthogonal representation of $\pi_1(M)$ and the flat bundle associated with that representation (this convention will be used for the whole paper). Here, ζ is the zeta function for the Laplacian $\Delta_q = (d\delta + \delta d)$ defined on the space of q-forms with values in V, where d is the exterior differentiation and δ is the formal adjoint of d. The zeta function of a Laplacian is defined as follows. If $\lambda_1 \leq \lambda_2 \leq \dots$ are positive eigenvalues of Δ_q listed in the increasing order counting multiplicities. Then the series

$$\zeta_q(s) = \sum_{i>0}\frac{1}{\lambda_i} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t\Delta_q}dt$$

is defined for s with Re$s > 1$ provided that $H^q(M,V) = 0$. This function of s has the analytic continuation to the entire complex plane with a continuous derivative at $s = 0$. Equivalently,

$$\tau_{\mathbb{R}}(M,V) = \prod_{0\leq q\leq n}(\det{}'(\Delta_q))^{(-1)^{q+1}q},$$

where $\det'(\Delta_q)$ is the regularized determinant of Δ_q defined as $\det'(-\Delta_q) = \exp(\zeta'_q(0))$.

We have the theorem of Ray and Singer :

Theorem 4. [RS1] Let M be a closed manifold and V be an orthogonal representation of $\pi_1(M)$ with the property that the cohomology of M with coefficients in V is trivial.

Then $\tau_{\mathbb{R}}(M,V)$ is metric independent.

Let V^* be the dual of V then if we consider the dual complex defined by the formal adjoint of the exterior differentiation, we have

$$\tau_{\mathbb{R}}(M,V) = \tau_{\mathbb{R}}(M,V^*)^{(-1)^{(n+1)}}$$

which implies the vanishing result of Ray and Singer when the dimension of M is even.

Theorem 5. When the dimension of M is even, $\tau_{\mathbb{R}}(M,V) = 1$, or equivalently, $\log\tau_{\mathbb{R}}(M,V) = 0$.

In particular, the real analytic torsion $\tau_{\mathbb{R}}(M,V)$ vanishes for complex manifolds. But for a complex manifold the deRham complex has a canonical subcomplex, namely, the Dolbeault complex. In the sequel [RS2] to their paper about the real analytic torsion, Ray and Singer defined complex analytic torsion $\tau_{\mathbb{C}}(M,V)$ of a n-dimensional complex manifold M as the regularized determinant of the Dolbeault complex

$$0 \longrightarrow \Omega^{0,0}(M,V) \xrightarrow{\bar{\partial}} \Omega^{0,1}(M,V) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{0,n}(M,V) \longrightarrow 0,$$

using the $\bar{\partial}$-Laplacian $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ and its zeta function. Here V is the vector bundle associated with a unitary representation of $\pi_1(M)$. The complex analytic torsion is also known as the holomorphic torsion. Ray and Singer showed that the ratio between two complex analytic torsions corresponding to different unitary representations of the fundamental group is independent of the Hermitian metric chosen, if all the cohomology of the Dolbeault complexes vanish.

As we have seen in the real case, a certain duality property for the de Rham complex gives the vanishing of the real analytic torsion when the dimension of the manifold is even. The dual map in the de Rham complex is given by the Hodge-$*$ operator which is defined using the volume form. In the complex manifold case, we can define a similar duality map if there exists a parallel holomorphic volume form. Hence this idea leads us to the condition of Calabi-Yau manifolds.

Recall that a compact Kähler manifold M is called a Calabi-Yau manifold if it has trivial canonical line bundle, $K_M = \mathcal{O}_M$. By the theorem of Yau [Y1], such a manifold admits a unique Ricci flat Kähler metric in each Kähler class. Equivalently, such a manifold has $SU(n)$ holonomy where $n = \dim_{\mathbb{C}} M$. Using Bochner arguments, we can show that the unique(up to scalar multiplications) holomorphic n-form Ω on M is parallel. Therefore, $\Omega \wedge \bar{\Omega}$ is a constant multiple of the volume form on M and we call Ω a holomorphic volume form on M.

Now we prove a complex analog of the Ray-Singer vanishing theorem for real analytic torsions:

Theorem 1. If M is an even dimensional Calabi-Yau manifold, then

$$\log\tau_{\mathbb{C}}(M,V) = 0,$$

where $\tau_{\mathbb{C}}(M,V)$ is the complex analytic torsion for M with coefficient in the unitary flat bundle V.

Proof. Let M be a Calabi-Yau manifold with complex dimension $2n$. Let $H_q^\lambda = \{\phi \in \Omega^{0,q}(M,V) | \Delta_{\bar\partial}\phi = \lambda\phi\}$, $E_q^\lambda = \{\phi \in H_q^\lambda | \bar\partial\phi = 0\}$, ${E'}_q^\lambda = \{\phi \in H_q^\lambda | \bar\partial^*\phi = 0\}$.

Then H_q^λ is an orthogonal direct sum of E_q^λ and ${E'}_q^\lambda$, $H_q^\lambda = E_q^\lambda \oplus {E'}_q^\lambda$.

Let $N_q^\lambda = \dim E_q^\lambda$ and ${N'}_q^\lambda = \dim {E'}_q^\lambda$. By the isomorphism $\frac{1}{\sqrt{\lambda}}\bar\partial : {E'}_q^\lambda \longrightarrow E_{q+1}^\lambda$(with the inverse $\frac{1}{\sqrt{\lambda}}\bar\partial^*$),

$$\dim H_q^\lambda = N_q^\lambda + {N'}_q^\lambda = N_q^\lambda + N_{q+1}^\lambda = {N'}_q^\lambda + {N'}_{q-1}^\lambda.$$

Hence,

$$\begin{aligned}\zeta_q(s) &= \sum_{\lambda\neq 0} \frac{1}{\lambda^s}(N_q^\lambda + {N'}_{q+1}^\lambda) = \sum_{\lambda\neq 0} \frac{1}{\lambda^s}(N_q^\lambda + N_{q+1}^\lambda)\\ &= \sum_{\lambda\neq 0} \frac{1}{\lambda^s}({N'}_q^\lambda + {N'}_{q-1}^\lambda).\end{aligned}$$

and

$$\sum_{q=0}^{2n}(-1)^q q\zeta_q(s) = \sum_{q=1}^{2n}(-1)^q \sum_{\lambda>0}\frac{1}{\lambda^s}N_q^\lambda = \sum_{q=0}^{2n-1}(-1)^{q+1}\sum_{\lambda>0}\frac{1}{\lambda^s}{N'}_q^\lambda.$$

Let Ω be the holomorphic volume form on M. Then we have an isomorphism $A : \Omega^{0,q} \longrightarrow \Omega^{2n,q}$ given by wedging with Ω. Since V is hermitian, the dual Dolbeault complex is identified with the original complex. Hence we get the isomorphism $A^{-1} \circ \bar{*} : E_q^\lambda \longrightarrow {E'}_{2n-q}^\lambda$. Hence we have $N_q^\lambda = N_{2n-q}^\lambda$. Therefore,

$$\begin{aligned}\sum_{q=1}^{2n}(-1)^q \sum_{\lambda>0}\frac{1}{\lambda^s}N_q^\lambda &= \sum_{q=0}^{2n-1}(-1)^{2n-q}\sum_{\lambda>0}\frac{1}{\lambda^s}N_{2n-q}^\lambda = \sum_{q=0}^{2n-1}(-1)^q\sum_{\lambda>0}\frac{1}{\lambda^s}{N'}_q^\lambda\\ &= -\sum_{q=0}^{2n-1}(-1)^{q+1}\sum_{\lambda>0}\frac{1}{\lambda^s}{N'}_q^\lambda = -\sum_{q=1}^{2n}(-1)^q\sum_{\lambda>0}\frac{1}{\lambda^s}N_q^\lambda.\end{aligned}$$

Hence the summation is zero and the log of the complex analytic torsion given by

$$\log \tau_{\mathbb{C}}(M,V) = \sum_{q=0}^{2n}(-1)^q q\zeta_q'(s) \mid_{s=0}$$

vanishes.

□

In particular, if the manifold has a holonomy contained in $Sp(n)$, (namely a HyperKähler manifold) then the manifold has a trivial complex analytic torsion. In the next section, we will define the quaternionic analytic torsion for such manifolds. More generally, they are defined for manifolds with its holonomy contained in $GL(n,\mathbb{H})Sp(1)$, namely, quaternionic manifolds.

3. Quaternionic Analytic Torsions

3.1. **Basic Facts about Quaternionic Manifolds.** A 4n-dimensional manifold M is called an almost quaternionic manifold if there is a rank 3 subbundle $\mathcal{G}$ of $\mathrm{End}(TM)$ such that for each $x \in M$ there is a neighbourhood U of x over which $\mathcal{G}|_U$ has a basis $\{I, J, K\}$ of almost complex structures with $K = IJ = -JI$. Note that this is only a local basis. An example of this basis over an open subset of $\mathbb{R}^{4n}$ is, with the usual identification of tangent space of $\mathbb{R}^{4n}$ with the $\mathbb{R}^{4n}$ itself,

$$I = \begin{pmatrix} 0 & -1_n & 0 & 0 \\ 1_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -1_n \\ 0 & 0 & 1_n & 0 \end{pmatrix}, J = \begin{pmatrix} 0 & 0 & -1_n & 0 \\ 0 & 0 & 0 & 1_n \\ 1_n & 0 & 0 & 0 \\ 0 & -1_n & 0 & 0 \end{pmatrix}, K = \begin{pmatrix} 0 & 0 & 0 & -1_n \\ 0 & 0 & -1_n & 0 \\ 0 & 1_n & 0 & 0 \\ 1_n & 0 & 0 & 0 \end{pmatrix}$$

where 1_n represents the $n \times n$ identity matrix. In other words,

Definition 6. A real 4n-dimensional manifold is almost quaternionic if the structure group of its principal frame bundle can be reduced from $GL(4n,\mathbb{R})$ to $G = GL(n,\mathbb{H})Sp(1)$.

If M is a Riemannian manifold, the metric g is said to be compatible with the almost quaternionic structure $\mathcal{G}$ if

$$g(AX, AY) = g(X, Y)$$

for $X, Y \in T_xM_{\mathbb{C}}$ and $A \in \mathcal{G}_x$ such that $A^2 = -1$. Locally, we can construct a compatible metric g from any Riemannian metric g' by defining

$$g(X,Y) = \frac{1}{4}(g'(X,Y) + g'(IX,IY) + g'(JX,JY) + g'(KX,KY)).$$

for $X, Y \in TM_{\mathbb{C}}$. Actually, this definition doesn't depend on the choice of the basis $\{I, J, K\}$. hence g is defined globally.

Note that the space of all the metric compatible with the given quaternionic structure is contractible. From this fact, when we consider a variation of compatible metric, it is enough to look at only local variations.

Next, we discuss quaternionic manifolds.

Definition 7. A real $4n$-dimensional almost quaterionic manifold M is quaternionic if the principal frame bundle of M admits a torsion-free $GL(n,\mathbb{H})Sp(1)$-connection.

Note that the existence of a torsion-free G-connection is only a partial obstruction to the existence of an integrable G-structure.

If M is a quaternionic manifold of dimension $4n$, then its holonomy group is a subgroup of $GL(n,\mathbb{H})Sp(1)$ and the reduced frame bundle F consists of frames $u : \mathbb{H}^n \longrightarrow T_xM$ which are compatible with the quaternionic structure. Locally, F can be lifted to a principal $GL(n,\mathbb{H}) \times Sp(1)$-bundle $\tilde{F}$ which double-covers F. This bundle exists globally if $(-1,-1) \in GL(n,\mathbb{H}) \times Sp(1)$ acts as the identity. The obstruction to the global existence of the double cover $\tilde{F}$ is the vanishing of the cohomology class $w(M) \in H^2(M,\mathbb{Z}_2)$ defined as follows.

Let $[F]$ denote the element of the cohomology group $H^1(M;GL(n,\mathbb{H})Sp(1))$ corresponding to the principal $GL(n,\mathbb{H})Sp(1)$-bundle F over M.

¿From the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow GL(n,\mathbb{H}) \times Sp(1) \longrightarrow GL(n,\mathbb{H})Sp(1) \longrightarrow 0$$

we get a coboundary homomorphism

$$\delta : H^1(M;GL(n,\mathbb{H})Sp(1)) \longrightarrow H^2(M;\mathbb{Z}_2).$$

If we define $w(M) = \delta[F] \in H^2(M;\mathbb{Z}_2)$, then the class $w(M)$ is the obstruction to lifting F to a principal $(GL(n,\mathbb{H}) \times Sp(1))$-bundle $\tilde{F}$.

¿From the lifting to a principal $GL(n,\mathbb{H}) \times Sp(1)$ bundle, we get the decomposition $TM_{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C} = E \otimes H$, where E is the vector bundle associated to the standard representation of $GL(n,\mathbb{H})$ on $\mathbb{H}^n = \mathbb{C}^{2n}$, and H is the vector bundle associated to the standard representation of $Sp(1)$ on $\mathbb{H}^n = \mathbb{C}^{2n}$.

For $q \leq 2n$, we denote

$$A^q = \bigwedge^q E \otimes S^q H$$

, where S^qH denotes q-th symmetric power of H and define D to be the differential operator defined as d followed by the projection onto A^q. Then by the work of S. M. Salamon [S1], we get an elliptic complex

$$0 \longrightarrow A^0(M) \xrightarrow{D} A^1(M) \xrightarrow{D} \cdots \xrightarrow{D} A^{2n}(M) \longrightarrow 0.$$

Remark. If the class $w(M) = 0$, then we have two vector bundles E and H. However, the existence of $A^q(M) = \bigwedge^q E \otimes S^qH$ and the elliptic complex

$$0 \longrightarrow A^0(M) \xrightarrow{D} A^1(M) \xrightarrow{D} \cdots \xrightarrow{D} A^{2n}(M) \longrightarrow 0,$$

do not depend on $w(M)$ being zero or not. This complex will be called the quaternionic complex. We define the cohomology group for this complex $H^q_A(M) = H^q(A^*(M), D)$ for any quaternionic manifold.

Remark. (The four dimension case) Notice that every Riemannian four manifold M determines a quaternionic manifold because of $SO\,(4) = Sp\,(1)\,Sp\,(1) \subset GL\,(1,\mathbb{H})\,Sp(1)$. Moreover, $w(M)$ equals to the second Stiefel-Whitney class of M, $w_2(M)$. Therefore $w(M) = 0$ is equivalent to $w_2\,(M) = 0$, namely M is a Spin manifold. This identification follows from the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{Z}_2 & = & \mathbb{Z}_2 & = & \mathbb{Z}_2 \\
\downarrow & & \downarrow & & \downarrow \\
Spin(4) & = & Sp(1)\times Sp(1) & \subset & GL(1,\mathbb{H})\times Sp(1) \\
\downarrow & & \downarrow & & \downarrow \\
SO(4) & = & Sp(1)Sp(1) & \subset & GL(1,\mathbb{H})Sp(1)
\end{array}$$

Furthermore, we have $E = S^-$ and $H = S^+$, where $S^\pm$ are the positive/negative spinor bundles over M and the quaternionic complex is the same as the self-dual complex:

$$0 \to \Omega^0(M,V) \xrightarrow{d} \Omega^1(M) \xrightarrow{P_+ d} \Omega^2_+(M) \to 0.$$

This is because of the canonical identifications $T^*_M \otimes_{\mathbb{R}} \mathbb{C} = S^- \otimes S^+$, $\Lambda^2_+ T^*_M = \mathfrak{su}(S^+)$ and the fact that $S^\pm$ are $SU(2)$ bundles.

In dimension four, people also call an Einstein manifold with quaternionic structure a quaternionic manifold.

Remark. There are two important classes of quaternionic manifolds which had been extensively studied in the literatures. Namely, they are the quaternionic Kähler manifolds (holonomy inside $Sp(n)Sp(1)$) and the HyperKähler manifolds (holonomy inside $Sp(n)$).

3.2. **The Definition of Quaternionic Analytic Torsion.** In this section, let the manifold M be a compact quaternionic manifold. In other words, we have the tensor product decomposition $TM_{\mathbb{C}} = E \otimes H$ of the complexified tangent bundle of M.

If we define the Young diagram λ_q for $q > 2n$ to be the diagram obtained by joining $2n \times 1$ diagram with λ_{q-2n}, we can also define A^q for q bigger than $2n$. Using these, we can define another complex

$$0 \longrightarrow A^{2n} \xrightarrow{d} \cdots \xrightarrow{d} A^{4n} \longrightarrow 0.$$

Consider the pairing,

$$A^q \otimes A^{4n-q} \xrightarrow{\alpha} \bigwedge^{4n} TM_{\mathbb{C}} = S_{\lambda_{4n}} E \otimes S_{\lambda'_{4n}} H.$$

Since we have the volume form vol on $\bigwedge^{4n} TM_{\mathbb{C}}$ and the inner product on A^q given by g, we can define $\gamma : S_{\lambda_q} \longrightarrow S_{\lambda_{4n-q}}$ by

$$\alpha(a \otimes \gamma b) =< a, b > vol$$

for $a, b \in A^q$. We can represent the pairing using Young diagrams.

$$
\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline \vdots\\ \hline \vdots\\ \hline q\\ \hline\end{array}
\otimes
\begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & q\\ \hline\end{array}
\otimes
\begin{array}{|c|c|}\hline 1 & 2n+1\\ \hline 2 & 2n+2\\ \hline \vdots & \vdots\\ \hline \vdots & 4n-q\\ \hline \vdots & \\ \hline 2n & \\ \hline\end{array}
\otimes
\begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & 2n\\ \hline 2n+1 & 2n+2 & .. & 4n-q\\ \hline\end{array}
$$

$$
\longrightarrow
\begin{array}{|c|c|}\hline 1 & 2n+1\\ \hline 2 & 2n+2\\ \hline \vdots & \vdots\\ \hline \vdots & 4n-q\\ \hline \vdots & 1\\ \hline \vdots & \vdots\\ \hline 2n & q\\ \hline\end{array}
\otimes
\begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & & \cdots & & & 2n\\ \hline 2n+1 & 2n+2 & .. & 4n-q & 1 & .. & q\\ \hline\end{array}
$$

Note that $\gamma\gamma = 1$.

Let V be an orthogonal representation of the fundamental group $\pi_1(M)$. Then we can consider differential forms on M with values in V. Then we can define the quaternionic analytic torsion using this complex and the Laplacian $\Delta_q^D = D\gamma d\gamma + \gamma d\gamma D$. Denoting $\gamma d\gamma = \delta$, we can write $\Delta_q^D = D\delta + \delta D$.

Definition 8. If M is a quaternionic manifold and V is a flat vector bundle, then the quaternionic analytic torsion is defined by

$$\tau_{\mathbb{H}}(M, V) = \prod_{q\geq 0} (\det{}' \Delta_q^D)^{(-1)^{q+1}q/2},$$

where $\det{}' \Delta_q^D$ is the regularized determinant of Δ_q^D.

3.3. **Invariance of the Quaternionic Analytic Torsion.** Let M^{4n} be a $4n$-dimensional quaternionic manifold and g_u be a one parameter family of metrics on M compatible to the almost quaternionic structure parametrized by u. We denote $\dot\gamma = \frac{\partial}{\partial u}\gamma$ and $\alpha = \gamma^{-1}\dot\gamma = \gamma\dot\gamma = -\dot\gamma\gamma$. For simplicity we denote Δ_q^D as Δ_q. Then

$$
\begin{aligned}
\dot\Delta &= D\dot\gamma d\gamma + D\gamma d\dot\gamma + \dot\gamma d\gamma D + \gamma d\dot\gamma D\\
&= D\dot\gamma\gamma\gamma d\gamma + D\gamma d\gamma\gamma\dot\gamma + \dot\gamma\gamma\gamma d\gamma D + \gamma d\gamma\gamma\dot\gamma D\\
&= -D\alpha\delta + D\delta\alpha - \alpha\delta D + \delta\alpha D
\end{aligned}
$$

With repeated application of $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for bounded operators as in the Ray and Singer's work, we have

$$
\begin{aligned}
\mathrm{tr}(e^{t\Delta_q}\alpha\delta D) = \mathrm{tr}(e^{\frac{1}{2}t\Delta_q}\alpha\delta D e^{\frac{1}{2}t\Delta_q}) = \mathrm{tr}(\delta D d^{t\Delta_q\alpha}),\\
\mathrm{tr}(e^{t\Delta_q}\delta\alpha D) = \mathrm{tr}(D e^{t\Delta_q}\delta\alpha),
\end{aligned}
$$

$$\operatorname{tr}(e^{t\Delta_q} D\alpha\delta) = \operatorname{tr}(\delta e^{t\Delta_q} D\alpha).$$

Hence we have

$$\begin{aligned}
\frac{d}{du}\operatorname{tr}(e^{t\Delta_u}) &= t\operatorname{tr}(\dot{\Delta} e^{t\Delta_u}) \\
&= t\operatorname{tr}(-\alpha\delta D e^{t\Delta_q} + \alpha D\delta e^{t\Delta_{q+1}} - \alpha\delta D e^{t\Delta_{q-1}} + \alpha D\delta e^{t\Delta_q}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{q=0}^{2n}(-1)^q q\operatorname{tr}(\dot{\Delta} e^{t\Delta_q}) &= \sum_{q=0}^{2n}(-1)^q\operatorname{tr}(\alpha\Delta_q e^{t\Delta_q}) \\
&= \sum_{q=0}^{2n}(-1)^q\frac{d}{dt}\operatorname{tr}(\alpha e^{t\Delta_q})
\end{aligned}$$

Let ζ_1 and ζ_2 denote two zeta functions corresponding to two representations V_1 and V_2, and $\Delta^{(1)}$ and $\Delta^{(2)}$ denote corresponding Laplacians. Then the variation of the ratio between two quaternionic analytic torsions is given by

$$\begin{aligned}
\frac{d}{du}\log\tau_{\mathbb{H}}(M,V_1)/\tau_{\mathbb{H}}(M,V_2) &= \frac{1}{2}\sum_{q=0}^{2n}\frac{d}{du}(-1)^q q\left(\zeta_{1q}'(0) - \zeta_{2q}'(0)\right) \\
&= \frac{d}{ds}|_{s=0}\frac{1}{2}\frac{1}{\Gamma(s)}\frac{d}{du}\sum_{q=0}^{2n}(-1)^q q\int_0^\infty t^s\operatorname{tr}(e^{t\Delta_u})dt \\
&= \frac{d}{ds}|_{s=0}\frac{1}{2}\frac{1}{\Gamma(s)}\sum_{q=0}^{2n}(-1)^q\int_0^\infty\frac{d}{dt}\operatorname{tr}(\alpha e^{t\Delta_{(1)}} - \alpha e^{t\Delta_{(2)}})dt \\
&= \operatorname{tr}(\alpha(P^{(1)} - P^{(2)}))
\end{aligned}$$

where P denotes the orthogonal projection onto the space of harmonic forms corresponding to the Laplacian of proper degree.

Hence if the space of the harmonic forms is trivial, i.e., if the cohomology $H_A^q(M,V_1)$ and $H_A^q(M,V_2)$ of the quaternionic complexes are all trivial, then the variation is 0.

Here, we have used the fact that

$$\operatorname{tr}(e^{t\Delta_{(1)}} - \alpha e^{t\Delta_{(2)}}) = O(e^{-c/t}) \quad \text{as} \quad t\to 0$$

and that $\operatorname{tr}(\alpha e^{t\Delta_{(1)}} - \alpha e^{t\Delta_{(2)}})$ decreases exponentially since Δ is strictly negative, hence the integration in the second line in the above equation can be differentiated inside the integral sign.

Hence we have proven

Theorem 9. For a quaternionic manifold M the ratio

$$\tau_{\mathbb{H}}(M,V_1)/\tau_{\mathbb{H}}(M,V_2)$$

of two quaternionic analytic torsions corresponding to unitary representations V_1 and V_2 is independent of the choice of the compatible metric if the cohomology group H_A^* is trivial.

Remark. Recall that the Riemannian structure on M^4 determines a quaternionic structure and the quaternionic complex for M is the same as the self-dual complex. It is not difficult to see from the inclusion

$$\begin{array}{ccc} SO(4) & \subset & GL(1,\mathbb{H})Sp(1) \\ \| & & \| \\ Sp(1)Sp(1) & \subset & \mathbb{H}^\times \cdot Sp(1) \end{array}$$

that two Riemannian metrics g and g' determines the same quaternionic structure on M provided that they are conformal to each other, that means $g' = e^{2v}g$ for some smooth function v on M.

As a result, the regularized determinant of the self-dual complex of M depends only on the conformal class of the metric. In the next section, we shall see that the self-dual analytic torsion is always a conformal invariant quantity.

Recall that the real analytic torsion vanishes on complex manifolds and the complex analytic torsion vanishes on HyperKähler manifolds. It is natural to ask if the quaternionic analytic torsion vanishes for certain special class of HyperKähler manifolds. In fact to look for the corresponding vanishing result, we should consider analytic torsions defined for manifolds admitting *commuting* complex structures instead of anti-commuting complex structures as in HyperKähler manifolds. Manifolds admitting commuting complex structures are studied in recent years by some string theorists (For example, see Rocek [R]) as candidates for mirrors of rigid Calabi-Yau manifolds of dimension three. Examples of such manifolds includes the Hopf surface or products of the Hopf surface with any complex manifold. (Recall that the Hopf surface is $\mathbb{C}^{2*}/(z_1,z_2) \sim (2z_1,2z_2)$ with complex structure descended from $\mathbb{C}^2$.)

4. Self Dual Analytic Torsions

As we discussed before, the quaternionic analytic torsion for 4-dimensional quaternionic manifold is the same as the determinant of the self-dual complex of the manifold.

In this section, we shall define the self-dual analytic torsion for 4n-dimensional manifolds and show that the ratio between two analytic

torsions $\tau_{\mathrm{SD}}(M, V_1, g) / \tau_{\mathrm{SD}}(M, V_2, g)$ corresponding to two flat bundles V_1 and V_2 is a conformally invariant quantity. Then we study the Kähler surface in more details and compare various torsions on them.

4.1. **Definition and Invariance of Self-dual Analytic Torsion.** Let M be a $4n$-dimensional Riemannian manifold with metric g and V be the flat vector bundle associated with an orthogonal representation of $\pi_1(M)$. Then we have

Definition 10. The self-dual analytic torsion is defined by

$$\log \tau_{\mathrm{SD}}(M, V, g) = \sum_{q=0}^{2n-1} (-1)^{q+1} q \log \det (\triangle_q) - n \log \det (\triangle_{2n}).$$

where Δ_q is the Laplacian on q-forms on M with values in V.

Proposition 11. The self-dual analytic torsion is the regularized determinant of the following self-dual complex

$$0 \to \Omega^0(M, V) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{2n-1}(M, V) \xrightarrow{\sqrt{2}P_+ d} \Omega^{2n}_+(M, V) \to 0$$

where $P_+ = (* + id)/2$ is the projection to the space of self-dual forms.

This is an elliptic complex which can be regarded as half of the deRham complex. The other half is the anti-self-dual complex,

$$0 \to \Omega^{2n}_-(M, V) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{4n-1}(M, V) \xrightarrow{d} \Omega^{4n}(M, V) \to 0.$$

The indexes of these two complexes are respectively $\frac{1}{2}(\chi(M, V) + sign(M, V))$ and $\frac{1}{2}(\chi(M, V) - sign(M, V))$. Their sum is just the Euler characteristic of M with twisted coefficient V. In a similiar vein, we can define the *anti-self-dual analytic torsion* $\tau_{\mathrm{ASD}}(M, V)$. By the vanishing result of Ray and Singer, we get

$$\tau_{\mathrm{SD}}(M, V) \cdot \tau_{\mathrm{ASD}}(M, V) = 1.$$

Hence the anti-self-dual analytic torsion doesn't give any new information.

Proof. (of proposition) If we denote $\left(\sqrt{2}P_+ \circ d\right) \circ \left(\sqrt{2}P_+ \circ d\right)^*$ by $\triangle_+$, then it coincides with the ordinary Laplacian $\triangle_{2n}$ on self-dual forms. It is because

$$\begin{aligned} \triangle_{2n}\phi &= -d * d * \phi - *d * d\phi \\ &= -(* + 1)\, d * d\phi \\ &= -2P_+ d * d\phi \end{aligned}$$

where $\phi \in \Omega_+^{2n}(M)$ and on the other hand, we have $(P_+ \circ d)^* \phi = \delta\phi = -*d\phi$ because

$$\int \langle \psi, (P_+ \circ d)^* \phi \rangle = \int \langle (P_+ \circ d)\, \psi, \phi \rangle = \int \langle d\psi, \phi \rangle = \int \langle \psi, \delta\phi \rangle .$$

Hence

$$(P_+ \circ d) \circ (P_+ \circ d)^* \phi = -P_+ \circ d * d\phi = \frac{1}{2} \triangle_{2n} \phi$$

and we have

$$\begin{aligned} \operatorname{tr} e^{t\triangle_+}|_{\Omega_+^{2n}(M)} &= \operatorname{tr} e^{t\triangle_{2n}}|_{\Omega_+^{2n}(M)} \\ &= \operatorname{tr} e^{t\triangle_{2n}} \circ P_+|_{\Omega^{2n}(M)} \\ &= \operatorname{tr} e^{t\triangle_{2n}} \circ (*+1)/2|_{\Omega^{2n}(M)} \\ &= \frac{1}{2} \operatorname{tr} e^{t\triangle_{2n}}|_{\Omega^{2n}(M)} + \frac{1}{2} \operatorname{tr} e^{t\triangle_{2n}} *|_{\Omega^{2n}(M)} \end{aligned}$$

One can show that the variation of $\operatorname{tr} e^{t\triangle_{2n}} *|_{\Omega^{2n}(M,V)}$ under a conformal change of metrics is zero and hence a conformally invariant. In fact, it is a homotopy invariant, namely the signature of M with coefficients in V: ¿From McKean Singer's formula, we have $sign(M,V) = \sum_{q=0}^{4n} \operatorname{tr} e^{t\triangle_q} *|_{\Omega^q(M,V)}$. However, all non-middle dimensional contributions to the summation cancel each other and gives $sign\,(M,V) = \operatorname{tr} e^{t\triangle_{2n}} *|_{\Omega^{2n}(M,V)}$. Alternatively,

$$[d,\delta] : \Omega_+^{2n}(M,V) \to \Omega_-^{2n}(M,V)$$

is an isometry which preserves eigenspaces of Laplacian with non-zero eigenvalue. Therefore, $\operatorname{tr} e^{t\triangle_{2n}} *|_{\Omega^{2n}(M,V)} = \operatorname{tr} *|_{H^{2n}(M,V)}$ which is clearly the twisted signature of M with coefficients in V.

However, for the regularized determinant of an operator, we discard all the zero eigenvalues. So $\operatorname{tr} e^{t\triangle_{2n}} *|_{\Omega^{2n}(M,V)}$ does not contribute to the torsion. Hence we can see that the analytic torsion defined by the elliptic complex

$$0 \to \Omega^0(M,V) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{2n-1}(M,V) \xrightarrow{\sqrt{2}P_+d} \Omega_+^{2n}(M,V) \to 0$$

is given by

$$\log \tau_{\mathrm{SD}}(M,V) = \sum_{q=0}^{2n-1} (-1)^{q+1}\, q \log\det(\triangle_q) - n \log\det(\triangle_{2n})$$

□

Next we show that the self-dual analytic torsion is a conformal invariant quantity.

Theorem 12. For any two orthogonal flat bundles V_1 and V_2 over M with trivial cohomology group. The ratio of the two self-dual analytic torsion $\tau_{\text{SD}}(M, V_1, g) / \tau_{\text{SD}}(M, V_2, g)$ depends only on the conformal class of the metric g.

Proof. To compute the variation of the self-dual analytic torsion under conformal change of metrics $g(u)$, we need to know $\frac{\partial}{\partial u} \operatorname{tr} e^{t\triangle_q(u)}$. The following formulae (from Ray-Singer's paper) are valid for any change of Riemannian metric $\frac{\partial g}{\partial u}$

$$\frac{\partial}{\partial u} \operatorname{tr} e^{t\triangle_q(u)} = t \operatorname{tr} \left(e^{t\triangle_q(u)} \dot{\triangle}_q \right)$$

where $\dot{\triangle}_q = \frac{d\triangle_q(u)}{du} = \alpha\delta d - \delta\alpha d + d\alpha\delta - d\delta\alpha$ and $\alpha = *^{-1}\dot{*} = *^{-1}\frac{\partial *}{\partial u}$. Therefore,

$$\operatorname{tr}\ e^{t\triangle_q}\dot{\triangle}_q = \operatorname{tr}\ e^{t\triangle_q}\delta d\alpha + \operatorname{tr}\ e^{t\triangle_{q-1}}\delta d\alpha - \operatorname{tr}\ e^{t\triangle_q} d\delta\alpha - \operatorname{tr}\ e^{t\triangle_{q+1}} d\delta\alpha$$

because $d\triangle_q = \triangle_{q+1} d$ and $\delta\triangle_q = \triangle_{q-1}\delta$.

¿From Ray-Singer's paper, we have

$$\sum_{q=0}^{k} (-1)^q q \operatorname{tr} e^{t\triangle_q} \dot{\triangle}_q$$
$$= \sum_{q=0}^{k} (-1)^{q+1} \operatorname{tr}\ e^{t\triangle_q}\ \triangle_q\ \alpha + (-1)^k (k+1) \operatorname{tr}\ e^{t\triangle_k}\delta d\alpha + (-1)^{k+1} k \operatorname{tr}\ e^{t\triangle_{k+1}} d\delta\alpha$$

for any integer k. In their paper, they study the case when k is the dimension of M and define a differentiable invariant. For our purposes, we shall take $k = 2n-1$ when $\dim M = 4n$. To counter the error term $-2n \operatorname{tr}\ e^{t\triangle_{2n-1}}\delta d\alpha + (2n-1) \operatorname{tr}\ e^{t\triangle_{2n}} d\delta\alpha$ we need to consider

$$n \cdot \operatorname{tr}\ e^{t\triangle_2 n}\dot{\triangle}_2 n + \sum_{q=0}^{2n-1} (-1)^q q \operatorname{tr}\ e^{t\triangle_q}\dot{\triangle}_q.$$

Now

$$\operatorname{tr} e^{t\triangle_{2n}}\dot{\triangle}_2 n = \operatorname{tr}\ e^{t\triangle_{2n-1}}\delta d\alpha - \operatorname{tr}\ e^{t\triangle_{2n+1}} d\delta\alpha$$

because $\dot{\alpha} = 0$ on $\Omega^{2n}(M)$, under conformal change of metrics.

By applying above formulae, we get

$$n \operatorname{tr}\ e^{t\triangle_{2n}}\dot{\triangle}_{2n} + \sum_{q=0}^{2n-1} (-1)^q q \operatorname{tr}\ e^{t\triangle_q}\dot{\triangle}_q$$

$$= \sum_{q=0}^{2n-1} (-1)^{q+1} \operatorname{tr}\, e^{t\triangle_q} \triangle_q \alpha - 2n \operatorname{tr}\, e^{t\triangle_{2n-1}} \delta d\alpha + n(\operatorname{tr}\ e^{t\triangle_{2n-1}}\delta d\alpha - \operatorname{tr}\ e^{t\triangle_{2n+1}} d\delta\alpha)$$

$$= \sum_{q=0}^{2n-1} (-1)^{q+1} \operatorname{tr}\, e^{t\triangle_q} \triangle_q \alpha - n(\operatorname{tr}\ e^{t\triangle_{2n-1}}\delta d\alpha + \operatorname{tr}\ e^{t\triangle_{2n+1}} d\delta\alpha)$$

Since $\operatorname{tr}\ e^{t\triangle_{2n-1}}\delta d\alpha + \operatorname{tr}\ e^{t\triangle_{2n+1}} d\delta\alpha = 0$, we have

$$n \operatorname{tr}\ e^{t\triangle_{2n}} \dot{\triangle}_{2n} + \sum_{q=0}^{2n-1} (-1)^q\, q \operatorname{tr}\ e^{t\triangle_q} \dot{\triangle}_q = \sum_{q=0}^{2n-1} (-1)^{q+1} \operatorname{tr}\, e^{t\triangle_q} \triangle_q \alpha.$$

Therefore, for s with sufficiently large $\operatorname{Re} s$, we have

$$\frac{d}{du}\left[\sum_{q=0}^{2n-1} (-1)^q q\zeta_q(s) + n\zeta_{2n}(s)\right]$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty t^s \left(n \operatorname{tr}\ e^{t\triangle_{2n}} \dot{\triangle}_{2n} + \sum_{q=0}^{2n-1} (-1)^q\, q \operatorname{tr}\ e^{t\triangle_q} \dot{\triangle}_q \right) dt$$

$$= \sum_{q=0}^{2n-1} (-1)^{q+1} \frac{1}{\Gamma(s)} \int_0^\infty t^s \operatorname{tr}\, e^{t\triangle_q} \triangle_q \alpha dt$$

$$= \sum_{q=0}^{2n-1} (-1)^{q+1} \frac{1}{\Gamma(s)} \int_0^\infty t^s \frac{d}{dt} \operatorname{tr}(e^{t\triangle_q} \alpha) dt$$

¿From [RS2] it follows that

$$\frac{1}{\Gamma(s)} \int_0^\infty t^s \frac{d}{dt} \operatorname{tr}\ \left(e^{t\triangle_q^{(1)}} - e^{t\triangle_q^{(2)}} \right) \alpha dt$$

defines an analytic function of s which vanishes at $s = 0$ where $\triangle_q^{(i)}$ denotes the Laplacian on M with coefficient in V_i. We have

$$\frac{d}{ds}|_{s=0} \left[\frac{1}{\Gamma(s)} \int_0^\infty t^s \frac{d}{dt} \operatorname{tr}\ \left(e^{t\triangle_q^{(1)}} - e^{t\triangle_q^{(2)}} \right) \alpha dt \right] = \operatorname{tr}(P^{(1)} - P^{(2)})\alpha$$

where $P^{(i)}$ is the harmonic projection operator for $i = 0, 1$. ¿From our assumption, $P^{(i)} = 0$ and hence we have shown that

$$\frac{d}{du} \sum_{i=1}^{2} (-1)^i \left(\sum_{q=0}^{2n-1} (-1)^q q\zeta_q^{'}(0, V_i) + n\zeta_{2n}^{'}(0, V_i) \right) = 0$$

That is $\tau_{\mathrm{SD}}(M, V_1, g) / \tau_{\mathrm{SD}}(M, V_2, g)$ is independent of u. Hence, we have our theorem. $\square$

Remark. Since $\alpha = 0$ on middle dimensional forms, we get

$$\operatorname{tr}|_{\Omega^{2n}(M,V)} \left(P^{(1)} - P^{(2)}\right) \alpha = 0$$

Therefore, it is enough for us to assume that $H^k(M, V_i)$ are zero for $k < 2n$ for the result to hold.

Remark. Notice that if a closed $4n$-dimensional manifold M is a complex manifold, then the complexification of the self-dual complex has a subcomplex, namely the Dolbeault complex:

$$\begin{array}{ccccccccc}
0 \to & \Omega^0(M)_{\mathbb{C}} & \xrightarrow{d} & \cdots \to & \Omega^{2n-1}(M)_{\mathbb{C}} & \xrightarrow{\sqrt{2}P_+d} & \Omega^{2n}_+(M)_{\mathbb{C}} & \to & 0 \\
& \| & & & \cup & & \cup & & \\
0 \to & \Omega^{0,0}(M) & \xrightarrow{\bar\partial} & \cdots \to & \Omega^{0,2n-1}(M) & \xrightarrow{\bar\partial} & \Omega^{0,2n}(M) & \to & 0.
\end{array}$$

All these inclusions are trivial except possibly the last one which again can be checked directly (without assuming Kählerian property of M).

Remark. Similiarly if a closed $4n$-dimensional manifold M is a quaternionic manifold, then the complexification of the self-dual complex again has a subcomplex, namely the quaternionic complex:

$$\begin{array}{ccccccccc}
0 \to & \Omega^0(M)_{\mathbb{C}} & \xrightarrow{d} & \cdots \to & \Omega^{2n-1}(M)_{\mathbb{C}} & \xrightarrow{\sqrt{2}P_+d} & \Omega^{2n}_+(M)_{\mathbb{C}} & \to & 0 \\
& \| & & & \cup & & \cup & & \\
0 \to & A^0(M) & \xrightarrow{D} & \cdots \to & A^{2n-1}(M) & \xrightarrow{D} & A^{2n}(M) & \xrightarrow{D} & 0.
\end{array}$$

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References

[B1] J.-M. Bismut, H. Gillet, C. Soulé, *Analytic Torsion and Holomorphic Determinant Bundles I.*, Commun. Math. Phys. 115, 49-78 (1988)

[B2] J.-M. Bismut, H. Gillet, C. Soulé, *Analytic Torsion and Holomorphic Determinant Bundles II.*, Commun. Math. Phys. 115, 79-126 (1988)

[B3] J.-M. Bismut, H. Gillet, C. Soulé, *Analytic Torsion and Holomorphic Determinant Bundles III.*, Commun. Math. Phys. 115, 301-351 (1988)

[RS1] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds.*, Advances in Math. **7** (1971), 145-210.

[RS2] D. B. Ray and I. M. Singer, *Analytic torsion for complex manifolds*, Ann. Math. 98 (1973), 154-177.

[R] M. Rocek, *Modified Calabi-Yau manifolds with torsion*, in *Essays on Mirror manifolds*, edited by S.T.Yau, International Press (1992).

[S1] S. M. Salamon, *Quaternionic Manifolds*, Comuicazione inviata all'Instituto nazionale di Alta Matematica Francesco Severi.

[S2] S. M. Salamon, *Quaternionic KählerManifolds*, Invent. math. 67, 143-171 (1982)

[Se] R. S. Seeley, *Complex powers of an elliptic operator*, Proc. of Symp. on Pure Math., AMS, 10 (1967)

[Sw] A. Swann, *HyperKähler and quaternionic Kähler geometry*, Math. Ann. 289, 421-450 (1991).

[Y1] S. T. Yau, *On the Ricci curvature of a compact Kähler Manifolds and the Complex Monge-Amperé equation I*, Comm. Pure Appl. Math., 31, 339-411 (1978)

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